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ABSTRACT

This paper analyzes the views about teaching mathematics held by eight preservice elementary teachers enrolled in a mathematics course that emphasized small group discussion and an exploratory approach to mathematics. Analysis of the students' diaries, interviews, and group discussions reveal a body of traditional views about teaching mathematics in clear contrast with the message of the course. Although students showed progress as reflective learners of mathematics, the effect of the experience as teachers is unclear. The paper addresses the need to work towards teacher education models that combine these experiences in mathematics with a successful transition into the school world. (Author)

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Prospective Elementary Teachers' Thinking About Teaching Mathematics

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Prospective Elementary Teachers' Thinking About Teaching Mathematics

Abstract

This paper analyzes the views about teaching mathematics held by eight preservice elementary teachers enrolled in a course that emphasized small group discussion and an exploratory approach to mathematics. Analyses of the students' diaries, interviews, and group discussions reveal a body of traditional views about teaching mathematics in clear contrast with the message of the course . Although students showed progress as reflective learners of mathematics, the effect of the experience as teachers is unclear. The paper addresses the need to work towards teacher education models that combine these experiences in mathematics with a successful transition into the school world.

Introduction

This paper is part of a larger study, the goal of which was to describe and analyze the understandings and beliefs about mathematics of these preservice teachers as they did, talked, and wrote about mathematics. The open approach to instruction that characterized the course, where students worked in small groups and were constantly talking about mathematics, led to a relaxed atmosphere in which the students often expressed their views and feelings about becoming teachers. In fact, their need to relate what they were doing in the course to their future teaching and to express their fears and concerns about their upcoming teaching career was so important that it became one theme of the study. This paper focuses on this theme by first presenting and discussing the main ideas about teaching mathematics of these eight students. These ideas reflected their own experience with school mathematics and were in conflict with the messages that the current course wanted to convey. The paper then turns to the implications from this study for teacher education in mathematics from the perspective of the recent recommendations by professional organizations and the current research in this area.

Background

As I became more involved in the mathematics courses for preservice elementary teachers, I felt that there appeared to be a gap between what these courses delivered and what the prospective teachers expected. They were concerned by issues of classroom management, motivation, and how to teach a subject that many of them did not enjoy and with which they felt inadequate. Solving problems, explaining their reasoning, exploring the whys behind the procedures memorized in their earlier schooling seemed less relevant to them. "What kind of mathematics experience should these prospective teachers be getting in their teacher education program?" became a key question for me.

The literature on beliefs about mathematics (Ball, 1988/1989; Cobb, 1985; Peck, 1984; Schoenfeld, 1985) and on teaching approaches centered on students' construction of meanings in mathematics via peer dialogues (Easley, 1988; Lampert, 1988; Yackel, 1987) constitute an important part of the framework for my study. The course where the study took place deviated

from what had been the students' experience with mathematics courses, and challenged their beliefs about what it means to do, learn, and teach mathematics. Through tasks intended to create cognitive conflict and a discussion-centered approach where the students were to take responsibility for their own learning and were constantly pushed to reflect on their actions, I was able to gain access to their mathematical world.

The Setting

I believe it is important for the reader to develop a feeling for the setting in which the study took place. It will help clarify what went on with the students and give more detail on part of the conceptual framework that guided this research.

The study took place during the summer offering of the course "mathematics for elementary education majors." The course met for two hours everyday, five days a week for eight weeks. There were eight students enrolled--five seniors, one junior, and two graduate students, all female. I was the instructor for that course. In order to get the students to engage in doing and talking about mathematics, I organized the classroom as a mathematics laboratory. We moved to the science education laboratory which gave us lots of room to move around, tables for the students to work in groups, and a place to keep the manipulatives and any other appropriate materials, so that they were always available. Furthermore, the computer room was right next door, which made it easy for the students to go back and forth.[1]

I kept lecturing to a minimum. My usual approach was to present them with a series of tasks and let them work on these. The students sat in small groups, usually two groups of four. Within those groups, sometimes they worked in pairs for a while, and then the four of them shared their ideas. Sometimes, all the students worked on a specific task, such as discussing an algorithm that they had never seen before. Other times, several small groups worked on different questions.

I wanted the students to participate actively in their own learning experience. I wanted them to learn to value their own thinking and to achieve intellectual independence (Confrey, 1983; Yackel, 1987). By creating cognitive conflict situations (Fujii, 1987), by introducing the surprise factor (Cobb, 1987), I challenged the students' beliefs and pushed them to consider their ideas--something that many of them had probably never done. The students were encouraged to advance their own ideas and to construct their understanding of mathematics. Peer dialogues, student-student interactions in small groups facilitated this meaning construction (Bishop, 1985; Lampert, 1988; Weinberg & Gavelek, 1987). The students usually felt more comfortable

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[1] This helped the interviewing process. At times I had most of the students working on Logo, while I remained in the classroom working with two to three students.

exposing their ideas to their peers than to me, and they were more willing to give and take criticism among themselves than with me (Easley, 1988).

Lampert's (1988) description of a teaching approach inspired by Lakatos and Polya captures the spirit that I tried to have in my course. Lampert contrasts that approach to that followed by most school mathematics programs. The latter are characterized by remembering and following the rules given by the teacher, having the answers ratified by the teacher or other source of authority, using widely accepted and unchallenged rules, formulas and facts. On the other hand, her approach to mathematics instruction has the students participating in the mathematics discourse by conjecturing, challenging each other's ideas, defending them, refuting them. As she says, "In their talk about mathematics, reasoning and mathematical argument--not the teacher or the textbook--are the primary source of an idea's legitimacy" (p. 439). She also indicates that the roles and responsibilities of teachers and students, as well as the kinds of activities, need to be reconsidered and changed in order to challenge the conventional assumptions about doing mathematics. In her conclusion Lampert writes:

> I assumed that changing students' ideas about what it means to know and do mathematics was a matter of immersing them in a social situation that worked according to different rules than those that ordinarily pertain in classrooms, and then respectfully challenging their assumptions about what knowing mathematics entails. (p. 470)

My goal was to develop a feeling in these students that they could do mathematics, that they could develop their own ways of tackling a task. I often challenged their ideas by asking them to further explain these or by offering counterexamples. But more important, in some of the group discussions, this challenging and arguing came from the peers themselves. That was especially rewarding since I wanted them to develop a questioning attitude towards their learning of mathematics. As Confrey (1983) says, in a constructivist philosophy of teaching, "the emphasis is always on what a student believes and why she believes it. Questions are answered with questions" (p. 235).

Method

The goal of my research was to illustrate and analyze the understanding and beliefs about mathematics of eight preservice elementary teachers as they did and talked about mathematics. I did not have a specific set of questions that I was trying to answer. Rather, I wanted to gain an understanding of the students' mathematical world and I wanted them to tell me about it. The teaching approach served my research goal.

The method of research for this study falls within the boundaries of qualitative inquiry (Erickson, 1986; Schatzman & Strauss, 1973). It shared many of the characteristics of the ethnographic research tradition. However, this study is not as holistic as an ethnography would be (Eisenhart, 1988).

The Students

Prior to this study, I had noticed that the mathematics background of the students in the summer tended to be weaker, but also more homogeneous than that of the students during a regular year. Also, the fact that this was the last time that this course was being offered[2] may have had an effect on the kind of students enrolled. To start with, the enrollment was even smaller (8) than the already typically low numbers for summer offerings. Quite often students who took this course in the summer had been putting it off for as long as possible. Many of them had told me that they took it in the summer as their only course so that they would have these eight weeks "to work only on math and get it over with." Two of the students in the study had tried to take the course earlier but had dropped it. Five of the eight were seniors and if they had not taken this course that summer, it might have delayed their graduation. Five students had not taken any mathematics course since high school. Two students were in their mid-thirties; the others were in their early twenties.

Data Collection

To gain an understanding of what seemed rather complex, the students' mathematics world, I needed to look at it from a variety of perspectives. Due to the complexity of the situations, triangulation was an important aspect in the data collection for this study [3]. Thus, I used several sources of data: observations, informal conversations, written homework, essays and diaries, audiotaped interviews with one or two students, and audiotaped small group discussions. Appendix I contains a brief description of each of these sources of data.

Types of Tasks

I devised three types of tasks to address my research goal:

Problem-solving type tasks. These were tasks in which the students worked on what I expected to be challenging problems. They were questions which did not necessarily relate to the specific topic we were working on in class.

Tasks based on "things they have always known." Through these tasks I was aiming to get the students to be inquisitive about pieces of elementary mathematics that they probably had been using for many years. These students were likely to be teaching many of these in the near future, yet what was their understanding?

Tasks aimed at creating cognitive conflict. Most of these tasks presented scenarios where a student (usually an imaginary child) had done some mathematics which was likely to spur discussion among the students. These tasks ranged over alternative algorithms for the four

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[2] From then on, the five-hour credit course was to be replaced by two three-hour credit courses.

[3] Furthermore, through triangulation I tried to minimize the possible effect of role conflict, since I was both instructor and researcher in this study (Shirk, 1973).

arithmetic operations, an erroneous procedure yielding a correct answer, or conflict situations that occurred among the students themselves.

Appendix II contains the tasks that are central to the discussion on the students' ideas about teaching.

Data Analysis

In analyzing the data I followed a sequence similar to that described by Frank (1985/1986) for her dissertation. I proceeded in four stages. The first stage took place along with the study itself. It involved daily reflection on what had happened that day in class and how it tied to my research goal. My main concern at this stage was to look at the data in terms of possible gaps, unclear instances, issues worth pursuing, so that I could develop new tasks if necessary, or go back to the students for clarification. A very important component of this stage was the research log I kept throughout the study, in which I recorded my impressions and reflections on the method of research, on the students, and on myself as a researcher.

A second stage in the analysis of the data started shortly after the summer course ended. This was an organizational stage: getting all the data together. I did a complete transcription of all the interviews and small group discussions. I also did some preliminary analysis of the transcripts, but mostly I read them over and over so as to make them a part of me. I also read all the written work that I had collected from each student and wrote down a report for each of them in which I commented on each task in their written homework. As I was doing this, I kept track of any regularities both for a given student and across students. At the end of what I view as a stage of familiarization with the data, some themes started to emerge. I came out of this experience with a first framework on which to build my discussion.

The third stage brought me back to the literature, especially in the area of "Models for Mathematical Behavior." Three of these models gave me broad categories around which to build my analysis (Buchanan, 1984/1985; Charles, 1985; Frank, 1985/1986). Organizing the data into classes and finding relationships (Schatzman & Strauss, 1973) was a gradual and sometimes rather painful process. The complexity of what I was trying to understand seemed to grow the more I dwelt on it. I went through several coding stages with overlapping or non-parallel categories coming up. By the end of this stage several themes were taking shape, one of them being the students' views on teaching mathematics.

In the fourth stage I focused on writing. I include this stage in my description of the data analysis because the writing I did then helped me refine further my analysis. The pieces still did not all fit. I felt I needed a hard copy in front of me containing a record of what I had done until then. Writing really forced me to organize my thoughts and start narrowing down the issues.

The Students' Views on Teaching Mathematics

The students' experience with their own schooling had instilled in them some beliefs as to how things were supposed to work in school mathematics. For many, their experience with mathematics in school had not been a pleasant one[4], and they were especially sensitive to the issues of frustration, confusion and putting children off. The students were now in a mathematics course which, by deviating from what they had expected in terms of instruction, was challenging some of their beliefs. Three factors--the general characteristics of traditional school mathematics, their own experience with it, and the course they were now enrolled in--played important, and sometimes competing, roles in the students' talk about teaching.

The most prominent idea that all the students shared was that their role as teachers was to tell the children what to do. This is what most of them had experienced in their schooling and was consistent with their expectations even then as students. But, as I further analyzed their ideas on teaching, a somewhat contradictory situation emerged. On one hand, the students wanted the child to conform to the "school ways." This entailed ignoring this child's ideas in favor of what they felt were the accepted ways. It meant dismissing the child's agenda and imposing their own. On the other hand, and especially for some of the students, a matter of key importance was not to frustrate the child. This meant that they made sure they praised something in the child's work. How would these prospective teachers decide what to praise? How would they decide when to impose "the school way?" These two questions lead to my main question and concern: how ready were these prospective teachers to judge students' work in terms of mathematical creativity and validity? In the Basketball Problem[5], two students agreed with a student's work who had reached the right answer via a faulty reasoning. Several students seemed oblivious to the mathematical creativity exhibited in students' work (Nathan's Division, Kye's Subtraction).

Perhaps it was their insecurity and a sense of their lack of knowledge that prompted the students to want to tell children what to do. As Donna said:

> It's scary to go into the classroom with the idea of letting the children go in different directions and me following them.

How and What to Teach

The students had fairly strongly held ideas as to how and what to teach. In this section I will illustrate what their main ideas were.

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[4]Based on their essays on "A memorable event in my elementary school mathematics experience."

[5] See Appendix II for the tasks mentioned in this section (Basketball Problem, Nathan's Division, and Kye's Subtraction).

1. Teach linearly, one thing at a time, and practice:

> I agree that school overdoes it, but I am not sure about bombarding students' minds with such thing as subtraction and addition together. It is one thing to do it to college students, but kids? I can see many confused and defeated. [Ann]

Some of the students did not seem to be aware of the importance of teaching so as to bring out the connections in mathematics. They were concerned that "mixing topics" would lead to children's confusion. The students had an idea of a linear sequencing in teaching in which, for example, one first teaches addition, then subtraction, then multiplication, and then division. Not only was the separation in topics important, but the order of delivery too. One would expect then, that since the order seemed to be important, the students had some ideas about the linking process. For example, why should addition come before subtraction? Though the students did know some of the connections (e.g., "multiplication as repeated addition"), these connections were often rather weak (for example, multiplication as only repeated addition).

"Practice makes perfect", Lisa and Ann wrote in their diaries. The idea of practice was present in their thinking about both, teaching and learning:

> I can see that division is complex and children need to be taken through it slowly and practice it to get the "hang" of it. [Ann]

> I find number theory confusing, especially the proofs, but it helps if I practice it. [Vicky]

I also think that practice has a place in mathematics learning, as long as it goes in hand with understanding. It is useful to be able to perform certain algorithms in a automatic mode, so that one can think about other things in the problem. In my working with these students, I found myself at several points trying to focus on meaning (for example, through manipulatives, or concrete situations, or through focus on the derivation process), while the students appeared more interested in practicing the pencil and paper aspect of it (the algorithm).

2. Showing, telling the student how to do something was evident all along in their comments on children's work (e.g., many "I would tell the child that..."). But, it was especially evident in their teaching to their peers. On several occasions, some students had to "teach" something to some of their peers. Quite often the "this is how you set it up and just solve it" prevailed. For example, in theprocess of solving one problem they had to find "what percentage of 32 is 100." Betsy pushed Lisa to write down the equation ($100 = 32x$) even though Lisa insisted that she did not understand why:

> Betsy: That's it, and just solve it. Don't even look at the problem; solve that algebra problem.
> Lisa: It still confuses me though, but

Betsy: But just solve that algebra problem. ... You don't even really have to look at it in terms of percentages.

3. Teaching skills was a priority in their agenda :what is their concept of understanding?

> Although I really agree with practicing problem-solving and discussion in the classroom, I still feel that skills are essential and need to be a part of math curriculum. I guess, what I'm trying to say is that I feel that a balance of the two is essential. [Betsy]

This was Betsy's comment after I took all the students to a DIME[6] session. Several of the students expressed their concern about how to make sure that the children were still learning the basic skills in the teaching approach that the teachers in DIME followed. The students were firm believers in the importance of teaching rules and skills. As Joyce wrote in her essay at the end of the course[7]: "I will also be sure to teach skills along with the discovery learning." Joyce was very clear on this all through the course:

> There is nothing wrong with teaching them the skills and teaching them to, you know, invert and multiply and all that; there is nothing wrong with that, but then once they have it in their heads, maybe you can show them why it works.

I think that part of the reason they wanted to teach the rule and leave the why behind, was that they themselves had a very hard time in exploring some of these mathematical concepts with diagrams, rods, and other concrete aids. They found it very difficult to make sense out of these memorized rules, and were not sure it was worth it. Joyce's comment below arises from her own difficulty when working on illustrating $8\frac{1}{2} \times 6\frac{1}{4}$:

> Maybe after they [the children] have worked with it, you can show why that works by doing this [pointing at the diagrams], once they understand, once they have the rule in their head.

What did Joyce mean by "understand" in the above excerpt? Maybe she had in mind some kind of procedural understanding. Maybe she wanted the students to first become very proficient at performing this kind of operation, then have them explore reasons for the rule. What was Lisa's idea of understanding in the excerpt below where she reflects on her experience on teaching algebra to Carol?

> There are many concepts which I understand perfectly, however I don't know why it is the way it is.

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6 DIME (Dialogues In Methods of Education): the teachers in this group use what I view as a constructivist approach to teaching which involves peer-dialogues and having the children discuss and devise their own methods to tackle the task. The tasks are meaningful situations, rather than symbolic unrelated computation exercises. Respect for the children's' ideas is, in my opinion, the governing rule in their teaching.

7 Topic: Your views about mathematics, and about teaching and learning mathematics

She knew how to do the algebraic manipulations but she could not explain to Carol the reasons for what she was doing. As teacher educators, I think we should explore what students mean by "understanding". Lisa may have been using the term in a rather narrow interpretation, because as the rest of the excerpt shows, she did show considerable insight in what it means to understand something:

> When I went to explain algebra it was difficult to explain why you do things a certain way. I found myself actually searching for meaning in the math instead of just the right answer. In fact, I actually became a learner also.

After having worked in class on justifying the procedure for multiplication of fractions in which they used, among others, the Cuisenaire rods, Ann wrote in her diary:

> It was very intriguing to use the rods, but somewhat confusing. I understand how we did it, but wonder how children will understand it. I feel that it would be more meaningful to teach students the rule when multiplying fractions, you multiply the denominators to get the denominator and numerators to get the numerator. Later you could give the students the manipulatives to determine the answer.

What did Ann mean by meaningful? In fact, below is some of what Ann wrote in her journal from her position as learner. I believe it shows quite a contrast with her thinking as a teacher.

> I cannot get over how I have accepted that you inverted and multiply to divide a fraction without even questioning or being taught why we do it.
>
> As a learner, I have always been taught to memorize equations, formulas, or steps to solve problems. I never knew or questioned the "why" behind the solutions.

I think that both Ann and Joyce were having difficulty seeing the point of going through these rather lengthy explorations about the meanings of these rules, when in their experience, all they had ever needed was to use those rules. This is consistent with an approach to mathematics instruction in which getting answers fast is rewarded, that is, an approach in which product is the important aspect and in which using the taught rules to produce an answer is the predominant pattern. The excerpts below, from two of the students' diaries corroborate this:

> I never really thought much about math problems. Growing up, I was shown a formula, plugged in numbers, and got praised if I calculated the correct answer. I never really looked at the answer and said, "But why is this the answer?" I just accepted it as right and went on to the next problem. [Betsy]
>
> It also made me realize that I was looking for quick, easy solutions without really being able to explain how or why they worked. [Joyce]

These students were exploring, probably for the first time, the why's behind many of the procedures they had used over and over in their schooling. They found it a hard task. One of the teachers in Shaw and Jakubowski's (1991) study said that relational understanding was important but should come later, maybe at college level. She expressed this idea after having tried to explain the why's behind the traditional algorithm for subtraction to her sixth grade students. The researchers write:

> She [the teacher] said the students already knew how to do subtraction and teaching them why it worked was unnecessary. She continued by saying that a relational understanding was good, but it should come later, much later. ... [she] was convinced that students in her mathematics classes should learn the "how-tos" in mathematics.(p. 15)

4. Efficiency seemed to be a driving force in some of the students' thinking about teaching. The students' concept of efficiency, as I infer it from their comments and their teaching, was "get it done and fast." Along this line of thought, telling the student what to do served their idea of efficiency. For example, here is Joyce's comment about a former roommate:

> By the end of the semester she [Joyce's roommate] finally got into fractions and she was just totally blown away by, she could not understand it, and I'm like, don't try to understand it [laughing], just do it, and she couldn't get that.

This comment reflects a pervasive trend in school mathematics: a certain content has to be covered, and it seems to be expected that not everybody will understand everything, but at least they have to be able to perform the procedures. It is not clear to me how much of this approach to teaching rested on a belief that it was more "efficient," and how much on the fact that the students did not know how to explain the issue in question. Their pseudo-mathematical sounding explanations, or their simple repeating the steps in a procedure, as being their explanation, lead me to believe that it was their own lack of understanding. Joyce was probably the most outspoken on the issue of getting things done at the expense of understanding (even though she herself gave evidence of understanding much of the mathematics we did).

[talking about the procedure to divide fractions]
Joyce: You inverted the wrong one; see, this is the one you invert.
Carol: Why?
Joyce: Because, that's what you are dividing by; if I try to explain it like my math teacher did, I'm going to screw you up even more.

[trying to illustrate division of fractions]
Joyce: If you have 3/5 of a pie and want to divide it by 11/15 of a pie (laugh) or something; but that's sort of, I mean to try and explain it, is even more confusing than doing it.

Betsy's comment below shows another angle to their goal of efficiency:

> Knowing how to do a simple problem many ways is wonderful. However, I think you should be able to recognize the fast, easy way to do a problem, so that you can still perform on timed, standardized tests (such as the ACT test).

Reactions to Children's Work

Among the tasks the students worked on, there were some in which they had to comment on a child's work. My main interest was on the students' understanding of the mathematics involved in these tasks. But they often also added comments which related to their ideas on teaching. In this section I will focus on these comments in order to get some insight as to how the students may react to their students' work.

1. Surprise, disbelief. Donna, talking about Nathan's Division said:

> I can understand this works wonderful, but, how, I just can't see in reality a child coming up with, well, maybe a gifted child.

Though none of the other students were ever so explicit as Donna was, several of them conveyed the same idea in both Nathan's Division and Kye's Subtraction. Donna had already expressed her surprise at children doing things by themselves when she referred to her semester in England:

> I told the children (6-7 years) I wanted them to make crystal shapes with the materials provided.... That was the only instruction I gave and I continued around the room to help the language and art groups. To my surprise, the shapes that the children created were brilliant.

The fact that some of the students were surprised at children coming up with their own ways to approach a task was to be somewhat expected. These students had had practically no experience teaching children (probably even less listening to children). Also, judging from their comments on their own schooling, the thought of being creative in mathematics seemed foreign to them. Their generally limited experience with children doing mathematics on top of their own experience as learners may well have given them the idea that the "typical" child could not possibly do mathematics by him or herself and should be shown how to by the teacher. The thought that only "special" children could do creative mathematics was shared by several of the students as they made comments along the lines of "this method would be confusing to other children." For example, Vicky wrote about Kye's subtraction:

> It seems a very simple way for Kye to understand subtraction. It could be a confusing method for another child because you are dealing with negatives.

This view is also present among Ball's (1988/1989) students as they talked about essentially the same task. Several of Ball's students were surprised that a second grader would think of doing subtraction in this way. Like the students in my study, they too were concerned that this

method would confuse other children. As Ball writes "Implicit was the notion that if a second grader did really come up with this, the child would be an unusual and advanced student" (p. 286-287).

Stevenson (1987) points out that in the American tradition, the belief seems to be that success depends basically on ability. If one takes this belief to an extreme, Stevenson says "children of low ability will not achieve regardless of how hard they work" (p. 31). As Papert (1980) says, these popular beliefs make us classify ourselves as "bundles of aptitudes and ineptitudes, as being 'mathematical' or 'not mathematical,' 'artistic' or 'not artistic,'..." (p. 8). Such classifications can be rather destructive as one comes to believe that there is hardly anything one can do about one's "limitations"--that this is just the way things are. "These beliefs are often repeated ritualistically, like superstitions. And like superstitions, they create a world of taboos; in this case taboos on learning" (Papert, 1980, p. 42).

2. Confusion/fun. Ideas with which the students had difficulty, they would say that the children would find confusing or difficult. On Kye's subtraction, Vicky said:

> You have to be very careful in your adding in bases which some children just wouldn't get.

The fact is that she was having difficulties using Kye's method in bases other than ten. Referring to base conversion without going through base ten, Lisa wrote:

> I am completely lost; I think children would be lost also.

The implicit message seems to be: "If something confuses me, it is going to be confusing for the children also." Paralleling this, was the idea, "I find this fun, children would also enjoy it." I think it is important to examine teachers' ideas about "confusing" and "fun" because the teaching implications may be: don't teach what is "confusing," do teach what is "fun." These prospective teachers might label something as confusing, failing to see its mathematical value; or, they might view something as "fun," regardless of its lack of mathematical content.

In fact, an example along the lines of the latter actually occurred. At one point some of the students started talking about division by 0. Lisa then related a story, the beach analogy, which a former teacher of hers had told to help her/his students remember that one cannot divide by 0. The story went more or less like this: "when people go to the beach, some will wear something on the top, some will not, but everybody has to wear something on the bottom." Through this story the students would then remember that 0 divided by something is all right, but that something divided by 0 is not. All the students in my class loved the story, so much so that in the following semester two of them (neither of whom was Lisa) who were in my methods class, clearly remembered the story and told it to the rest of the class as something "neat."

The students labeled something as confusing or fun as a direct result of their understanding of the mathematics involved, as well as of their beliefs about what is proper in mathematics. These two factors may lead them to praise a child's work for the "wrong" reasons, or to fail to appreciate good mathematics, or to try to make the child conform to the school ways. These three issues will be the subject of the next sections.

3. Praise. Some of the students were very careful always to praise something in these imaginary children's work. Sometimes that praise was appropriate, sometimes it was not. But the point is that for some students, it seemed very important to write some positive comments about the children's work, even in these rather artificial situations. Ann was one such student; perhaps the main idea in her thinking about teaching was that of confidence. The child had to feel confident. As a teacher, her task was to make sure that she contributed to this confidence building. That involved praising some aspect of the child's work. Thus, for Kye's Subtraction, Ann wrote "Kye was very inventive" (even though she then tried to help him conform to the "traditional" way); for Nathan's division, she wrote:

> Nathan devised a very intelligent way to divide by breaking down the divisor.... I would commend his work, but point out the way to check as...

(She then proceeded to show how to check a division via multiplication, and how Nathan's division did not check.) On the Basketball Problem, she wrote:

> I would comment that the student was right to change all the shots to 20 since that was a common number (emphasize good) but when he did so...

(She then pointed out what the error was in his procedure.) Other students were also explicit about this idea of praising. Betsy, on Nathan's Division, wrote:

> As a teacher, you could help Nathan understand what is wrong with his technique (though he's on the right track) with manipulatives.

Many of the students had had experiences with mathematics in school which had, in their perception, undermined their confidence as learners in this area. They had experienced frustration as learners, and thus, avoiding frustration in their students seemed to be a very important aspect of their thinking about teaching. The question is: how were they planning to do that? Were they ready? In the Basketball Problem, two of the students thought that what the child had done (using an additive strategy in a ratio problem) was correct and praised his work. These two students' poor understanding of the mathematics involved in the problem led them to label an incorrect thinking strategy as "a brilliant idea" (Donna), or "creative, divergent, and insightful" (Carol).

The students' concern for preventing a child's frustration is a genuine and valid one. However, a poor understanding of mathematics may lead them to use praise inappropriately. As I illustrate next, it may also lead them to "fail to praise" a child's idea when they should.

4. Lack of appreciation. To be able to make sense out of someone's else thinking can be difficult. Quite often it involves stepping out of one's own way of looking at things in order to try to look at them from someone else's perspective. For some of the students whose range of ways of looking at things was rather limited, this may be really hard. The students were not used to seeing more than one approach to a problem. In fact this realization that there were many ways to approach a problem seemed to be a very powerful one, judging by their many comments on this in their diaries. This mode of thinking (i.e., "there is only one way to do something") was especially evident in the case of "elementary" mathematics topics, such as algorithms for the four arithmetic operations.

In my research log I kept bringing up the fact that I felt that the students did not share my enjoyment of certain pieces of mathematics. I enjoyed thinking about the division method presented in Duckworth (1987, adapted in Nathan's Division). I thought it helped one visualize division and reinforced the meaning. However, after Joyce had just explained this method and gave signs of understanding it, she said: "All he is, he's just regrouping with smaller numbers."

Joyce was somewhat puzzling in that she seemed to exhibit two different behaviors depending on whether she was doing mathematics or whether she was expressing her ideas about teaching. When doing mathematics, she would try different methods and she enjoyed discussions in which I tried to make them see connections among some of these methods. But, when talking about teaching, her main idea seemed to be "let's get it done."

Carol also made a similar comment on Nathan's Division task:

> But I mean, why? What's the big deal, all he is doing is splitting up a number, you know, I mean why are we doing this?

This comment should be viewed with caution. It may indicate a lack of appreciation of the child's work, but it may also be the result of Carol's growing discomfort with the course. I think that in most cases, what I labeled as "lack of appreciation towards creativity in mathematics" derived from the students' insecurity about what was really happening in a given mathematics problem. If the method under discussion "did not work," such as the first version of Nathan's Division, it was hard for the students to look at it in terms of its mathematical value. Some of them made vague comments of praise, but in general, they focused on pointing out what went wrong.

But even in the cases in which the child's method appeared to work, their comments did not show any sign of a "this is neat" reaction. Ann was the only student who commented on Kye's inventiveness though she was cautious about it (Kye' Subtraction).

> Kye was very inventive. I was unable to find error in his process, but in terms of time and room for error, it seems a very long process.

Joyce seemed to dismiss the value of Kye's procedure:

> Kye's procedure will always work because all he is doing is writing the process out in long hand.

There appeared to be two underlying factors that determined whether the students expressed surprise, showed a lack of appreciation, thought that what the child had done was confusing, or decided to praise some aspect of his work: first, the students' own usually rather narrow understanding, and second, their idea of school mathematics. The latter is what I discuss next.

5. Conforming to the school way. Most of the examples of the students' reactions to children's work presented in the previous sections show a unidirectional thinking in which the work is judged essentially by school standards. For example, when Ann said about Kye's Subtraction that "in terms of time and room for error it seems a very long process," or when Vicky referred to Nathan's Division as "he is doing double the work," they seemed to be bringing in their school notion of efficiency as meaning being fast. The standard algorithms for arithmetic do convey this idea of efficiency: they take little room (as compared to Kye's Subtraction, or Nathan's Division) and once one masters them, they are quite fast. However, other methods can be just as fast to those who master them. In fact, whether a person feels comfortable with a method is likely to have a lot to do with his or her performance.

Referring to Kye's Subtraction, Vicky said:

> I do believe that you could eventually convince him that learning to carry is easier and leaves less room for error.

Why did Vicky believe that carrying was easier? Perhaps it was just that she was used to that method? One of my goals in this course was to get the students to look at the contents of school mathematics with a critical attitude. It was not an easy task. It is hard to be critical of that which one does not understand but which seems to be accepted by a large group of people. The students were familiar enough with the school culture to have ideas about what might be expected from them when they received a teaching position. The comments that follow illustrate some of these ideas and the weight of school tradition on some of the students.

Donna, at the end of their discussion on Nathan's Division, said:

> I don't think teachers teach like this. When I did it they made you do it the long way.

I do not know what Donna had in mind by "the long way." I am guessing she meant the traditional long division algorithm. Vicky, referring to the same task, said:

> I would not want to teach somebody that; Nathan, he is a fifth grader? He's going to have a lot of problems.

Unfortunately, I did not pursue what she meant by her last statement. I suppose that in many typical school settings, a child like Nathan might well have some problems if he started deviating too much from the norm. In her exploration of division with elementary school teachers, Duckworth (1987) raises the issue of encouraging children to create their own methods versus imposing the standard ones. One of the teachers in Duckworth's study wrote: "the ultimate problem is to get the kids to understand the conventional representation of division" (p. 65).

In one of the frequent discussions by the students as to whether to teach different number bases to children, Carol said:

> But what about the addition facts that they need to get; what happens if you teach, say a bunch of first graders, base five, and then they go to another teacher; I mean there aren't workbooks that are, the workbooks are only in base ten.

Carol's concern about how one could teach different number bases if the workbooks "are only in base ten" makes me wonder about the likely influence that the textbook will play in her teaching.

When Carol and Joyce were working with manipulatives on subtraction in base four, Carol said:

> Wouldn't kids get confused? From left to right, wouldn't kids get confused? If I sat down with a group of kids and said, "Ok, this is how you do it, " and showed them from left to right, I would think that when you got to the real thing, that they would get upset or that they would be confused.

By the real thing Carol meant the standard pencil and paper algorithm for subtraction (which proceeds from right to left).

Some Reflections

At the beginning of the discussion on the students' ideas on teaching mathematics, I referred to a conflict in some of the students' thinking about teaching. On the one hand, they were concerned about lack of confidence and frustration in the child and wanted to prevent these from taking place. On the other hand, they felt the need to make sure the child learned what is expected, and in the way it is expected, as dictated by the school tradition. Was the conflict only in my mind? Were the students aware of it? My question is: did the students really believe in these ideas about teaching or did they want to believe in them? For example, did they really

believe that the standard algorithm for subtraction was actually best, or were they hiding their insecurity about other ways behind this belief?

From the time that I started working with preservice elementary teachers I had always had some trouble understanding why most of these students who, by their own testimony, had had a rather miserable experience with school mathematics, insisted on teaching in the very style that had failed with them. That Joyce and Betsy wanted to do so was not so surprising--the system had paid off enough to get them through their mathematics courses, and in fact in the present course their peers viewed them both as quite successful. Betsy described herself as being very good at learning formulas and memorizing and seemed contented with that. Joyce felt strongly that what was important was to get it done. Her statement "Don't try to understand it, just do it" seems to be a key aspect of her approach to teaching.

The students' wanting to conform to the way they were taught was probably to be expected after their over twelve years of schooling doing things a given way without questioning. Other researchers have pointed out the fact that preservice teachers receive many of the messages about how to teach while in their K-12 schooling (Ball, 1990b; Thompson, 1992). Furthermore, even if some of these prospective teachers do question their prior experience with school mathematics, they may lack alternative models to which to turn (Buchmann, cited in Ball 1990a).

This course did shake their perceptions about teaching and learning mathematics and some of the students gave signs of discomfort. Sometimes I wondered whether the experience was doing them more harm than good. At other times, I wondered whether they did not view the course as an isolated, somewhat odd experience and resisted it in an effort to preserve consisteny with themselves (Perry & Whitlock, 1954). Were the students clinging to ways of thinking imposed by their school experience in order to prevent the frustration that could result should they allow their beliefs to be shattered?

This study took place in a course of mathematics content for elementary education majors. One of the main goals of both the study and the course was to have the students explore their own understanding of mathematical ideas. Their thinking about teaching was not a priority in " my agenda", at least, not in the beginning. Yet, for these students their immediate access to the teaching world seemed to be their main concern. This is what I labeled as "their needs."

Their Needs

As a result of having the students talk and write about mathematics and about the course in general, I learned about what seemed to be a priority for some of them: their concern about becoming teachers. This concern made them try to view the course content in terms of how they would teach these ideas to their students, rather than to work on understanding these ideas themselves. One of the clearest examples of this was with their work on non-decimal bases. Some of the students were concerned about whether and how they would teach this to children,

while for me, this was never a question. My purpose in the work in non-decimal bases was to put these prospective teachers in an unfamiliar territory which would facilitate the exploration and discussion of facts they "have always known." Quite often the students would refer to children in their comments about ideas discussed in class. I am not sure whether the fact that they brought up the children or wrote about whether children would find a certain topic fun or hard, reflected a real concern in the students' thinking or was more a shield to hide their own thinking about the topic.

In their work with a group of practicing elementary teachers, Russel and Corwin (1991) remark that in the beginning, the teachers' discussion of the mathematics tasks focused on how to apply what they were doing to their own teaching. "This classroom focus acted as a barrier--and perhaps as a safety valve--to teachers' grappling with mathematics *for themselves*. ... As they gradually let go of immediate classroom application, they began to be captured by the pleasure of deep involvement in mathematics." (pp. 175-176).

But perhaps, for preservice teachers, it is not easy to "let go." Their upcoming teaching career presents too many unknowns (that experienced teachers may have already resolved) for them to "put it aside." Donna was very explicit about her need to make sense of this course in terms of her overall teacher preparation:

> I feel pretty good about the new way to subtract [Equal addition]. One thing that confuses me though is whether or not to introduce this to children. Marta, did you show us this because you thought it was neat (cool) or because you thought it would be good to show children both ways to subtract? In fact a lot of the things we do in math 202 confuse me. They confuse me because I do not know whether we are doing such problems for fun, mind refreshers, mind bogglers, mind frustrators or because it would be good to introduce such concepts (different bases, European subtraction, ...) to children? AM I justified for feeling this way; let's talk about it. Sometimes I don't know where we are going and I feel like this is just another course preparing me for a test. I guess maybe I'm expecting too much, but more discussion, more discussion about what we do, and how it relates to children would help. I think all 8 of us would agree. What do you think?
>
> I have been thinking a lot about this class and how it is going to prepare me for teaching. I realize that there is a methods course for math so this course is supposed to be a refresher course? Everything we are doing I have done at one time and I get frustrated when I cannot remember ways of solving problems.

Some students expressed their helplessness at the idea of not knowing what to do when confronted with students' questions. Mabel, when working on the fraction task in which a student had inverted the first fraction in a division problem, got visibly upset when realizing she did not know how to address this situation. She knew it was the second fraction that had to be inverted, but did not know why. She literally shouted "Why is it like that? I want to know." As I was pushing her to try to find out a way to help the student, she said, half jokingly:

> And I would say [to the student] "well, I don't know why because Marta didn't tell me."

Donna was the most outspoken about her fears of becoming a teacher. She expressed those right from the start, and I believe that as the course went along she became even more open about them. She would consistently turn almost every question I asked her into "what if a child came and asked me that, I don't know what I would tell him." The following situation summarizes the helplessness that I believe several of the students felt towards their upcoming teaching. In one of the interviews Donna and I were talking about a child's argument for $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$. As Donna tried to see how she could show the child what $\frac{1}{2} + \frac{1}{3}$ was, she said:

> Well, you know, I'd probably say, well this is the rule, you know, unfortunately; I don't want to do that because I'd rather show them, but I don't know any other way to approach it. So, I'll, you know, teach the way I was taught.

The fact is that these prospective teachers did have some needs with respect to their upcoming teaching career. Most of them were very close to start their student-teaching, and all had had some experience in the schools. Some of them expressed a concern that their college education was not preparing them to be teachers, that it did not deal with the reality of school, that most college instructors had never been in a classroom with children. I think one should view these comments cautiously: they may be more the result of what these prospective teachers keep hearing and they may be trying to use them as an excuse for their own failures to come to grips with the subject matter. Some research studies have pointed out that preservice elementary teachers are not satisfied with their preparation in mathematics in college because they do not see how it relates to what they will be doing in the school setting. They thus tend to base most of their acts on personal experience (Howsam, cited in Isenberg & Altizer-Tuning, 1984; Spitzer, cited in Warkentin, 1975). Shuard (1984) comments on the discrepancy that many preservice teachers find between what they are taught in college and what they actually see when they visit the schools, which tends to be more conservative. The fact that most teachers found their preservice education of little use is documented in the literature (Easley, 1979), and is sufficiently widespread for these prospective teachers to be aware of it. Whatever the reality may be, these prospective teachers' perceptions of the situation are likely to be what prevails. The students were interpreting the course in terms of their perceived reality of the school setting. As a result, changing their views about mathematics may be difficult, since they are unlikely to change if they do not see that the methods and ideas proposed in the college courses are in fact better than what is being currently used in the schools.

Conclusion

A course like the one reported in this study presented prospective teachers with a model that, in including many of the recent recommendations for mathematics instruction (MAA, 1991; NCTM, 1991; NRC, 1989), was very different from what they had experienced as students in school and from what they were likely to see in most of the schools they visited. Several research reports are coming out describing similar work with non traditional courses in mathematics for prospective teachers (Ball, 1990a; Raymond, Santos, & Masinglia, 1991; Simon, 1991). My impression is that we are going to see more and more of these courses in teacher education programs. What have we learned so far that could make these ongoing and future experiences more fruitful? All these studies, including this one, give hopeful signs of a development of a more reflective and critical approach in the participants' thinking about teaching and learning mathematics. As learners of mathematics in these courses, there seems to be progress in the ways they go about doing mathematics. Most of the students in this study gave clear signs of enjoying doing mathematics. They would get involved into the mathematical tasks and often could not let go. Their comments showed a growth as reflective learners about what it means to do mathematics: working in groups seemed to be a revolutionary idea for many of them, and they were ready to embrace it; they constantly contrasted their previous experience with the learning of mathematics to their current experience, and they gave evidence of great insight in their contrasting. As learners, they seemed to have moved along. But, as future teachers, what can be said about the effect of these experiences? Some of the students gave evidence of an inner struggle trying to make sense out of this course in view of their existing conceptions about teaching and learning mathematics. As learners, many of them were experiencing success. Vicky's statement very early in the course captures a feeling that I think many of the other students shared:

> I had fun today. In fact, I found myself looking forward to class. ... There is hope yet when I can legally use my methods to solve a problem.

As I pointed out earlier in the paper, there seemed to be a noticeable contrast in their comments depending on whether they were wearing the teacher's hat or the learner's hat. As future teachers, what sense were they making of this experience? Were they going to change their probably deeply rooted conceptions or were they going to try to assimilate the messages given in this course into their existing frameworks (Thompson, 1992)? The students' writing about teaching (especially in their final essay) contained samples that could be viewed as hopeful signs of their wanting to move towards a model of teaching more in accordance with the recent recommendations. Yet, I am skeptical. Were they trying to say the "right" thing? Was it mere rethoric? Many of these students were quite familiar with the education theories from their

education classes and they had the "proper" vocabulary to express themselves. But what did they really mean? I think that as mathematics educators we should explore what these students mean when they refer to "skills", "problem-solving", "understanding" and their talk about mathematics in general. We may be using the same terms with very different meanings in mind. I believe that prospective teachers receive too many conflicting messages from a variety of sources: their K-12 school experience, their field experience in the schools, their education courses, their content courses, their cooperating teachers, the community. I am afraid that they may grab onto the messages that make them feel more secure. The students' awareness of the school reality (at least in their perception) was very strong:

> Although I feel more comfortable about mathematics, the thought of teaching math scares me. In fact I think it scares a lot of elementary school teachers. Too many teachers rely on textbooks with the answers printed in black and white. Let us face the facts, most teachers teach this way because it is less scary for teachers.
> I want to teach both skills and problem solving techniques but I'm not sure how to go about it. I do not want to have to rely on a textbook to teach math, but at the same time I'm scared of approaching math from any other way. [Donna - Final essay]

Crosswhite (1987) describes the case of a teacher who quit after one year once she was faced with the reality of the school world and realized that she could not teach mathematics the way she wanted. From reading Crosswhite, I get the impression that this teacher was one I wish had stayed in the teaching profession. But, as Crosswhite points out, the reality of the schools did not leave much time or room for this teacher to be creative and to have her students pursue their own thinking and ideas.

I am concerned about what can be accomplished at the preservice level with the current models of teacher education. I would hope that courses like the one referred to in this paper help plant the seeds for these prospective teachers' development. But without follow-up and without what I think is most important, a coordinated and collaborative approach to their overall teacher education, I wonder if these seeds will ever germinate. As part of their preservice education, these prospective teachers should see these ideas and alternative approaches to teaching mathematics in practice, that is in actual classrooms with school-age students. Furthermore, even if some of the students may be willing to give a try to a different approach, I think that without a support structure in the school where they end up teaching, it is going to be very hard for them to succeed. The NCTM Professional Standards for Teaching Mathematics (working draft) called for a change in the current models for teacher education along the lines of this support structure I just mentioned:

> Few current models used by universities, schools, and communities involve working together to support new teachers. Often the "umbilical cord" is cut abruptly, and the constraints of the real world of schools overwhelm the fragile perceptions these new

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> teachers hold about what mathematics teaching and learning could be. The result is that many new teachers find it difficult to adapt what they have learned in their teacher preparation programs to the conditions in which they are teaching. These teaching standards specifically address the continuing responsibilities of universities, schools, and communities to give support to young teachers during their first years of teaching. (pp. 5-6)

I think that a next step is to develop and do research on models for teacher education centered around support structures for the prospective teachers. A priority in teacher education programs should be to bridge the gap between the perhaps idealistic view of school presented in college and the reality of schools. Without building up a sense of continuity to help prospective teachers make the transition from the university to the school world, teachers may not be able to profit fully from rich mathematical experiences in college, and may in fact resent them.

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single-person. Schoenfeld argues that when two persons work together on a problem, there is a greater chance for more talk, exchange of explanations and rationalization to take place. Also I think that having two students working together lessened the (perhaps) intimidating aspect of "being interviewed." To get the interview going I often tried to have two students who had done a certain problem differently on the homework, or two students where one was to explain something to the other.

Audiotaped small group discussions. The last and probably most important source of data was the tape-recording of their group discussions in class. These discussions were on a variety of mathematical tasks. On some occasions I first asked the students to write something and thus I had their written work along with the tape; but in most cases what I had was only the recording of their discussion and whatever notes I took. Usually the only structure to the discussion was that imposed by working on the task. The two groups were sometimes working on different tasks; sometimes there was social chit-chat going on; other times the students started talking about becoming teachers. The students did not seem to mind the presence of a microphone at the center of their table. Through these discussions I was able to analyze their understanding and beliefs about mathematics, their efforts to make sense out of a certain topic, their interactions. It gave me a rich array of situations in which the students were doing mathematics.

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Appendix I
Sources of Data

Observations. The students did most of their work in small groups sitting around tables. Often this work involved the use of manipulatives. I spent a large part of my time in class walking around, listening to and looking at what the students were doing. I brought along a small notebook where I recorded my observations. The main uses of this written record were to develop further tasks, or to keep track of the students' written work or their actions with the manipulatives, which would be lost on the tape recorder.

Informal conversations. These were conversations between the students and me, and they played an essential role in establishing rapport and helping build the atmosphere that I believe is necessary for this type of learning experience to succeed. They also helped me getting to know some of the students better. They usually took place either at the beginning or the end of each class period. But probably the longest and most fruitful conversations took place in the computer laboratory in the evening. Some of the students stayed there quite late talking to me about their degree, the class, their anxiety about becoming a teacher, or just more casual topics.

Written homework. The students handed in six homework assignments during the course. These assignments had four to five problems each, and the students usually had at least a week to work on them. The students had to explain what they did and I encouraged them to include their different attempts at solving a problem, whether "successful" or not. In addition to these homework assignments, the students took two tests and a final exam. As with the homework assignments, I followed up some of the questions on the tests with interviews.

Essays and diaries. The students wrote three essays during the course. The first one was to be handed in as early as possible in the first week. The topic was "A Memorable Event in my Elementary School Mathematics Experience". In the second half of the course, the students participated in a one day teaching-learning algebra experience. Four of the students were to be teachers of the other four students. They had to teach something of their choice in algebra. Afterwards, I asked the students to write a paper, the "teachers" on their experience, and the "learners" on theirs. The last week of the course the students wrote a paper on "Your Views About Mathematics, and About Teaching and Learning Mathematics."

The students also kept a diary during the whole course (one of the students quit writing it for the last third). In this diary I asked them to write about their thoughts and impressions as a learner of mathematics. I left it very open for the whole class, and elaborated and became more specific with each student as I read their diaries and wrote comments. These became a means of communication through which the students told me of their ideas about the course, and about learning and teaching mathematics. Some students wrote a lot, some not so much. Yet, they were all very regular in their entries, with an average of four per week. Since my "instructions" about the diary were rather vague, each student wrote about that which she felt like, which allowed me to get a better picture of each individual student. These diaries became a very important source of data, much more so than what I had initially expected.

Interviews with one or two students. The interviews were always on mathematics tasks, either especially designed for the interview, or the result of a homework assignment or a class event. The main purpose of these interviews was to find out more about the students' understanding of mathematics. As Davis (1984) describes, with each of these task-based interviews I ended up with an audiotape of the whole session, the students' written work, and my notes during the interview. The interviews usually took place during class time, while the other students were in the computer laboratory working on Logo. I interviewed each student at least twice. During the interviews, I used a combination of the talking aloud procedure and the clinical interview technique (Ginsburg, Kossan, Schwartz, & Swanson, 1983). At some points, in a few of the interviews, my role as an instructor took precedence over that of researcher, and the interview turned into a teaching session. At others, I just waited till the end of the interview to go over any problems that came up.

I always tried to interview two students at a time. In doing so, I followed Schoenfeld's (1985) recommendations concerning two-person protocols for problem-solving, rather than

Appendix II
Sample Tasks

[Basketball Problem] (Adapted from Hirabayashi, 1985)

A group of fifth graders were working on the following problem:
Three children are practicing basket-ball shooting; this is the table recording the results:

player	shots	successes
A	4	2
B	10	4
C	20	6

Question: who is the best player?
One of the fifth graders in his solution to the problem made the following table:

player	shots	successes
A	20	18
B	20	14
C	20	6

He then concludes that A is the best player.

You are asked to comment on this student's work.

[Nathan's Division] (Adapted from Duckworth, 1987)

Two sixth graders are having a disagreement as to how to divide; see if you can settle their differences. Nathan says that he has found another way to divide; for example, to divide 735 by 18, he claims that since 18 = 9 x 2, you can divide 735 by 2 and then divide the result by 9, as follows:

```
   367
2 | 735
    6
    13
    12
     15
     14
      1
```

and then do

```
    40
9 | 367
    36
     7
```

So that the answer to 735 ÷ 18 is 40 remainder 7

Glen does not agree with him because he says that if you do

```
   40
18 | 735
    72
     15
```

The answer is 40 remainder 15.

How do you explain the discrepancy in their results? Will it always be so? Why? Why don't the remainders agree?

[Nathan's Division (Revised)]

Nathan has been doing some further work on his procedure for division, and this is what he has come up with:

To divide 735 by 18 he does:
735+2 ---> Q = 367 and R = 1; where Q stands for quotient and R for remainder.
Then, 367 + 9 ---> Q = 40 and R = 7
So he says that 735 + 18 will yield a quotient of 40 and a remainder of 7 x 2 + 1 which is 15.

Let's consider more examples:
Suppose we do 735 + 9 ---> Q = 81 and R = 6 and then 81 + 2 ---> Q = 40 and R = 1; so, 735 + 18 ---> Q = 40 and R = 1 x 9 + 6 = 15.
To do 735 + 18 we could also divide by 6 and by 3; let's do it Nathan's way:
735 + 6 ---> Q = 122 and R = 3
122 + 3 ---> Q = 40 and R = 2
So, 735 + 18 ---> Q = 40 and R = 2 x 6 + 3 = 15

Let's do a different example, 44 + 15:
44 + 5 ---> Q = 8 and R = 4
8 + 3 ---> Q = 2 and R = 2
So, 44 + 15 ---> Q = 2 and R = 2 x 5 + 4 = 14

Nathan seems to be getting correct answers; is that so? If yes, why? Does his method make sense? Explain it.

[Kye's Subtraction] (Adapted from Davis, 1984)

Kye, a third-grader, "invented" this way of subtracting:

```
  73
 -58
  -5   ( 3 - 8 = -5)
+ 20   (70 - 50 = 20)
  15
```

So the answer is 15; let's try another one:

```
  235
 -189
   -4    (5 - 9 = -4)
+ -50    (30 - 80 = -50)
  100    (200 - 100 = 100)
   46
```

So, the answer is 46.

Will Kye's procedure always work? If yes, why? If no, why not?
Will Kye's procedure work in bases other than ten? Justify.

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